Problem 3.45

Find the position operator in the basis of simple harmonic oscillator energy states. That is, express

$$\langle n \,|\, \hat{x} \,|\, \mathcal{S}(t) \rangle$$

in terms of $c_n(t) = \langle n|\mathcal{S}(t)\rangle$. *Hint*: Use Equation 3.114.

Solution

The Position Operator: Method 1

Use the method of Example 2.5 on page 47 and express the position operator in terms of the promotion and demotion operators, $\hat{a}_+$ and $\hat{a}_-$, respectively.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)$$

Note that the promotion and demotion operators satisfy

$$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle.$$

Therefore,

$$\begin{aligned}
\langle n \,|\, \hat{x} \,|\, \mathcal{S}(t)\rangle = (\langle n|\hat{x}) \cdot |\mathcal{S}(t)\rangle &= \left(\hat{x}^\dagger |n\rangle\right)^\dagger \cdot |\mathcal{S}(t)\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)^\dagger\,|n\rangle\right]^\dagger \cdot |\mathcal{S}(t)\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_+^\dagger + \hat{a}_-^\dagger\right)|n\rangle\right]^\dagger \cdot |\mathcal{S}(t)\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_- + \hat{a}_+)\,|n\rangle\right]^\dagger \cdot |\mathcal{S}(t)\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_-|n\rangle + \hat{a}_+|n\rangle)\right]^\dagger \cdot |\mathcal{S}(t)\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right)\right]^\dagger \cdot |\mathcal{S}(t)\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\langle n-1| + \sqrt{n+1}\,\langle n+1|\right) \cdot |\mathcal{S}(t)\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\langle n-1 \,|\, \mathcal{S}(t)\rangle + \sqrt{n+1}\,\langle n+1 \,|\, \mathcal{S}(t)\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,c_{n-1}(t) + \sqrt{n+1}\,c_{n+1}(t)\right].
\end{aligned}$$

The Position Operator: Method 2

Alternatively, use the result of Problem 3.39 which says

$$\langle n \,|\, \hat{x} \,|\, n' \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\, \delta_{n,n'+1} + \sqrt{n'}\, \delta_{n',n+1} \right)$$

and note that the eigenstates of the harmonic oscillator are orthonormal. Doing so gives

$$\begin{aligned}
\langle n \,|\, \hat{x} \,|\, \mathcal{S}(t) \rangle &= \langle n \,|\, \hat{I}\hat{x}\hat{I} \,|\, \mathcal{S}(t) \rangle \\
&= \langle n | \left(\sum_{j=0}^{\infty} |j\rangle\langle j| \right) \hat{x} \left(\sum_{k=0}^{\infty} |k\rangle\langle k| \right) |\mathcal{S}(t)\rangle \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \langle n \,|\, j \rangle \langle j \,|\, \hat{x} \,|\, k \rangle \langle k \,|\, \mathcal{S}(t) \rangle \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \delta_{nj} \langle j \,|\, \hat{x} \,|\, k \rangle \langle k \,|\, \mathcal{S}(t) \rangle \\
&= \sum_{k=0}^{\infty} \langle n \,|\, \hat{x} \,|\, k \rangle \langle k \,|\, \mathcal{S}(t) \rangle \\
&= \sum_{k=0}^{\infty} \langle n \,|\, \hat{x} \,|\, k \rangle c_k(t) \\
&= \sum_{k=0}^{\infty} \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\, \delta_{n,k+1} + \sqrt{k}\, \delta_{k,n+1} \right) c_k(t) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{n} \sum_{k=0}^{\infty} \delta_{n,k+1} c_k(t) + \sum_{k=0}^{\infty} \sqrt{k}\, \delta_{k,n+1} c_k(t) \right] \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{n}\, c_{n-1}(t) + \sqrt{n+1}\, c_{n+1}(t) \right].
\end{aligned}$$

The Momentum Operator: Method 1

Here the aim is to find the momentum operator in the basis of simple harmonic oscillator energy states. Use the method of Example 2.5 on page 47 and express the momentum operator in terms of the promotion and demotion operators, $\hat{a}_+$ and $\hat{a}_-$, respectively.

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}_+ - \hat{a}_-)$$

Note that the promotion and demotion operators satisfy

$$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle.$$

Therefore,

$$\begin{aligned}
\langle n\,|\,\hat{p}\,|\,\mathcal{S}(t)\rangle &= (\langle n|\hat{p})\cdot|\mathcal{S}(t)\rangle \\
&= \left(\hat{p}^\dagger|n\rangle\right)^\dagger\cdot|\mathcal{S}(t)\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}_+ - \hat{a}_-)^\dagger\,|n\rangle\right]^\dagger\cdot|\mathcal{S}(t)\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}_+^\dagger - \hat{a}_-^\dagger\right)|n\rangle\right]^\dagger\cdot|\mathcal{S}(t)\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}_- - \hat{a}_+)\,|n\rangle\right]^\dagger\cdot|\mathcal{S}(t)\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}_-|n\rangle - \hat{a}_+|n\rangle)\right]^\dagger\cdot|\mathcal{S}(t)\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle\right)\right]^\dagger\cdot|\mathcal{S}(t)\rangle \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\,\langle n-1| - \sqrt{n+1}\,\langle n+1|\right)\cdot|\mathcal{S}(t)\rangle \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n}\,\langle n-1\,|\,\mathcal{S}(t)\rangle - \sqrt{n+1}\,\langle n+1\,|\,\mathcal{S}(t)\rangle\right] \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n}\,c_{n-1}(t) - \sqrt{n+1}\,c_{n+1}(t)\right].
\end{aligned}$$

The Momentum Operator: Method 2

Alternatively, use the result of Problem 3.39 which says

$$\langle n \,|\, \hat{p} \,|\, n' \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\,\delta_{n,n'+1} - \sqrt{n'}\,\delta_{n',n+1}\right)$$

and note that the eigenstates of the harmonic oscillator are orthonormal. Doing so gives

$$\begin{aligned}
\langle n \,|\, \hat{p} \,|\, \mathcal{S}(t) \rangle &= \langle n \,|\, \hat{I}\hat{p}\hat{I} \,|\, \mathcal{S}(t) \rangle \\
&= \langle n| \left(\sum_{j=0}^{\infty} |j\rangle\langle j|\right) \hat{p} \left(\sum_{k=0}^{\infty} |k\rangle\langle k|\right) |\mathcal{S}(t)\rangle \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \langle n \,|\, j\rangle\langle j \,|\, \hat{p} \,|\, k\rangle\langle k \,|\, \mathcal{S}(t)\rangle \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \delta_{nj}\langle j \,|\, \hat{p} \,|\, k\rangle\langle k \,|\, \mathcal{S}(t)\rangle \\
&= \sum_{k=0}^{\infty} \langle n \,|\, \hat{p} \,|\, k\rangle\langle k \,|\, \mathcal{S}(t)\rangle \\
&= \sum_{k=0}^{\infty} \langle n \,|\, \hat{p} \,|\, k\rangle c_k(t) \\
&= \sum_{k=0}^{\infty} i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\,\delta_{n,k+1} - \sqrt{k}\,\delta_{k,n+1}\right) c_k(t) \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n}\sum_{k=0}^{\infty} \delta_{n,k+1} c_k(t) - \sum_{k=0}^{\infty} \sqrt{k}\,\delta_{k,n+1} c_k(t)\right] \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n}\,c_{n-1}(t) - \sqrt{n+1}\,c_{n+1}(t)\right].
\end{aligned}$$